# The development of turbulent boundary layers with negligible wall stress

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In a recent paper, Stratford has described a turbulent boundary layer with continuously zero wall stress and has developed a theory to describe the flow based on two assumptions. The first is that the flow in the outer part of the layer is affected only by the original Reynolds stresses during the initial development, and the second is that flow in the equilibrium layer close to the wall is determined by the pressure gradient and is independent of upstream conditions. In this paper the same assumptions are used, but more careful consideration of their limitations has led to the elimination of some inconsistencies in the original work and to a theory that gives a better description of some of the experimental results. The principal results are: (i) a criterion for zero wall stress in an adverse pressure gradient of sufficient strength, (ii) the form of the pressure distribution for a self-preserving flow with zero stress, (iii) the mean velocity distribution in this flow, (iv) an estimate of the constant in the 'square-root' velocity distribution for flow near a wall with zero stress.

## 1. Introduction

The prediction of the course of development of a turbulent boundary layer in an arbitrary adverse pressure gradient is a problem of considerable practical importance in engineering, and a number of methods have been developed, some of which lead to reasonably accurate results. The more recent and successful methods take notice of the existence of a 'constant-stress' layer near the surface and approximate to the velocity profile by a combination of a logarithmic wall profile and an outer profile, but the combining of the two parts has been an arbitrary procedure justified only by results. Stratford (1959*a*), in a very interesting paper, points out that, in a boundary layer subjected to a severe pressure gradient over a short distance, the outer profile is determined by the initial profile and the pressure rise, and he goes on to develop a very simple criterion for zero stress by combining this observation with a deduction of the wall profile for zero stress. The significance of this work is not that it provides a more accurate prediction of pressure rise to separation than other methods (in fact, it is less accurate), but that it is founded on a physically acceptable model of the boundary-layer motion and that results derived from the model should have very wide validity if the nature of the underlying approximations is kept in mind. Although this work marks a step forward in the replacement of assumptions made for mathematical convenience by assumptions conforming to physical reality, the assumptions that the initial profile has a power-law form, that the junction of the wall profile and the outer profile is necessarily smooth, and that the wall profile can extend to the outer edge of the layer, fall short in this respect and cause some inconsistency between the theory and the experimental results. This paper presents an alternative version of the theory that leads to a more consistent and complete description of the turbulent boundary layer with zero wall stress (Stratford 1959*b*) and that is consistent with present knowledge of self-preserving boundary layers. It is, in fact, a commentary on the original work of Stratford.

## 2. Notation

Turbulent boundary layers on a flat, smooth surface are considered using Cartesian co-ordinates to describe the flow. The axes are chosen so that $0x$ is in the surface and in the direction of mean flow (assumed two-dimensional) and $0y$ is at right angles to the surface. Then

$U, V, 0$ are the components of mean velocity
$u, v, w$ are the components of the velocity fluctuation
$P$ is the pressure at the wall $y = 0$
$\tau_0$ is the shear stress at the wall
$U_1$ is the free-stream velocity
$\dfrac{dP}{dx} = -U_1\dfrac{dU_1}{dx}$ is the longitudinal pressure gradient
$U_0$ is the free-stream velocity at $x = x_0$, the beginning of the adverse pressure gradient
$c_f = 2\tau_0/v_0^2$ is the local friction coefficient, immediately upstream of $x_0$
$c_p = 1 - U_1^2/U_0^2$ is the pressure recovery coefficient
$\gamma = \tau_0^{\frac{1}{2}}/(KU_0)$ is the local friction parameter
$K$ is the Kármán constant
$K_0$ is the constant in the velocity profile for zero stress
$R = [\tau_0(x)/\tau_0(x_0)]^{\frac{1}{2}}$
$\delta = \nu\tau_0^{-\frac{1}{2}}\exp(\gamma^{-1} - A)$ is the boundary-layer thickness at $x_0$
$\eta = y/\delta$ is a non-dimensional co-ordinate
$\nu_T$ is the effective eddy viscosity for the outer part of a self-preserving boundary layer
$R_s = \dfrac{1}{\nu_T}\displaystyle\int_0^\infty (U_1 - U)\,dy$ is a flow constant, equivalent to a Reynolds number based on the eddy viscosity

Validity of the boundary-layer approximation is assumed.

## 3. The two-layer model

It is well known that the turbulent flow very close to a solid boundary is very different in nature from the flow near the outer edge of a boundary layer, and that one reason for the difference is that the wall flow has a high rate of energy dissipation and so is nearly in a state of equilibrium determined by local conditions while the outer flow has a low rate of dissipation and its structure is dependent on conditions far upstream of the point of observation. In general, the parts of the flow which have these properties in the purest forms can develop almost independently (e.g. entering upon a region of strong pressure gradient, modification of the wall flow is nearly instantaneous while the outer flow is modified at a much more gradual rate), and the first useful approximation for the description of a turbulent boundary layer is to represent it as the juxtaposition of an inner wall flow and an outer edge flow, neglecting the blending region within which the dissipation is neither large nor small. In the real flow, interaction between the two parts of the flow takes place in the neglected blending region, and this can be described in the model by imposing suitable conditions at the junction. If the layer is self-preserving, this distinction between the layers is expressed by assuming the velocity distribution in the outer layer to be that arising from constant eddy viscosity in the outer flow and a logarithmic distribution of velocity in the inner layer (Townsend 1956*a*; Clauser 1956), and this leads to satisfactory results if the junction is assumed smooth. If the layer is not self-preserving, the assumption of constant eddy viscosity must fail (in the outer layer, 'turbulent fluid' behaves like a visco-elastic fluid with a comparatively long relaxation time), and the junction conditions require special consideration.

Consider the changes in a turbulent boundary layer as it flows into a region of severe adverse pressure gradient after a period of development in favourable or zero gradient, and choose the origin so that $x_0$, the start of the pressure gradient, is equal to the distance from the leading edge at which a boundary layer developing in zero gradient would have the same friction coefficient. Initially, the stress gradients are of order $\tau_0/\delta$, and the criterion of severity is that accelerations due to the pressure gradient should be large compared with $\tau_0/\delta$, i.e.

$$c_f \ll \frac{\delta}{x_0}\left(x_0\frac{dc_p}{dx}\right), \tag{3.1}$$

and, since $\delta/x_0$ is of the order of $c_f^{\frac{1}{2}}$, this requirement is satisfied if

$$x_0\frac{dc_p}{dx} \gg c_f^{\frac{1}{2}}. \tag{3.2}$$

In a flow satisfying (3.2), the total head is very nearly constant along streamlines except in regions where stress gradients have increased very considerably. This will happen first very near the wall, where the rate of dissipation of turbulent energy is so large that any reduction of the energy supply by retardation of the mean flow leads almost instantaneously to lower levels of turbulent intensity and to lower Reynolds stresses. Further from the wall, the rate of dissipation is less, and an appreciable change in Reynolds stress will take longer and the stress gradient will not become comparable with the pressure gradient so close to the start of the expansion. Sufficiently far downstream, supposing the layer to be still attached, the Reynolds stresses will be modified on all streamlines and will be determined more by the pressure distribution than by the initial values. These effects are illustrated in figure 1, which shows the variations of Reynolds stress with stream function at various distances from the beginning of the adverse gradient.

It is useful to distinguish three stages of development. In the *initial* stage, Reynolds stresses have been modified only in a thin layer close to the wall and this layer forms an equilibrium layer within which the flow is substantially determined by the local wall stress and pressure gradient. The remainder of the flow has been acted on only by the pressure field and, to a small extent, by the initial Reynolds stresses. Further retardation causes the region of modified Reynolds stress to form an appreciable part of the whole flow, and then only a part of the modified region is in a state of wall equilibrium and has a structure independent of upstream conditions. In the last stage of development, Reynolds stresses over the whole layer have been affected by the retardation, and it may happen that flow becomes *self-preserving* and independent of the initial flow. If this occurs, past experience suggests that Reynolds stress and mean velocity

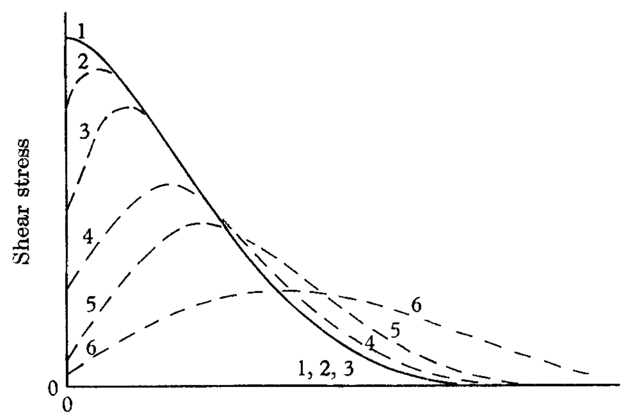


FIGURE 1. Distributions of shear stress in a boundary layer entering a region of strong adverse pressure gradient. (1) Initial distribution at $x = x_0$. (2) and (3) Initial stage of development. (4) and (5) Intermediate stage: modified stresses no longer confined to equilibrium layer. (6) Final stage: stresses modified everywhere.

gradient will be related through a coefficient of eddy viscosity in the outer layer. It follows that there are means of inferring the distributions of mean velocity both in the inner and outer layers if the layer is either in the initial stage of development or in a stage of self-preserving development, and a complete description is possible if the position and nature of the junction can be determined.

Two necessary conditions that must be satisfied at the junction are continuity of mean velocity and of Reynolds stress, as is easily seen by remembering that the neglected blending region has properties intermediate between those of the inner and outer layers. If mean velocity were discontinuous in the model, the mean velocity gradient in the blending region would be abnormally large, which would lead to abnormally high (or low) rates of production of turbulent energy and to abnormal Reynolds stresses. It follows that the inner and outer velocity distributions, when produced, intersect in the blending layer. Similarly, a discontinuity of Reynolds stress would lead to abnormal acceleration of the mean flow and the distributions of Reynolds stress must also intersect in the blending region and be continuous in the two-layer model. In the initial stage of development, these two conditions are sufficient to determine the nature of the junction and any further condition is redundant, but a third condition must be imposed in the self-preserving stage of development. The reason for this is that stress gradients are then not negligible and, if the flow obeys the equation of mean motion, continuity of velocity implies continuity of stress. In the initial stage, Reynolds stress gradients are negelected in the equation of mean motion and, to this approximation, Reynolds stress and mean velocity are capable of independent variation. The third junction condition, applicable only to self-preserving flows, is usually taken to be continuity of mean velocity gradient, equivalent to continuity of effective eddy viscosity.

## 4. Wall equilibrium for zero wall stress

The concept of the equilibrium layer was first developed for the constant-stress layer, but it is capable of considerable extension to flow along any boundary on which the mean velocity is specified. Sufficiently close to any solid boundary, the convective terms in the averaged equations of motion are negligible compared with the stress terms, and the equations reduce to

$$\frac{\partial \overline{uv}}{\partial y} = -\frac{\partial P}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} \tag{4.1}$$

or, in integrated form,

$$\tau = -\overline{uv} + \nu \frac{\partial U}{\partial y} = \frac{\partial P}{\partial x} y + \tau_0. \tag{4.2}$$

So the distribution of shear stress depends only on the local parameters, $\tau_0$ and $\partial P/\partial x$, and it may be inferred that the motion within a thin layer for which (4.2) is valid depends only on these parameters and on the fluid viscosity. The detailed justification of this assumption depends on the possibility of also neglecting convective terms in the equation for the turbulent kinetic energy, which implies that the turbulent motion is in a state of energy equilibrium. Within the fully turbulent part of such a layer, it is well known that

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\left(\frac{\tau_0^{\frac{1}{2}} y}{\nu}\right) + A\right] \tag{4.3}$$

if $\tau_0 \gg (\partial P/\partial x)\, y$; but Stratford (1959*a*) has shown that, if $\tau_0 \ll (\partial P/\partial x)\, y$,

$$\frac{\partial U}{\partial y} = \frac{1}{K_0}\left(\frac{\partial P}{\partial x}\right)^{\frac{1}{2}} y^{\frac{1}{2}}, \tag{4.4}$$

and that, for zero wall stress

$$U = \frac{2}{K_0}\left(\frac{\partial P}{\partial x} y\right)^{\frac{1}{2}} + C\left(\nu \frac{dP}{dx}\right)^{\frac{1}{3}} \tag{4.5}$$

(Here $K_0$ is an absolute constant, expected to be similar in magnitude to the Kármán constant $K$.) The condition that convective terms in the equation of mean motion are negligible is that

$$\left| \int_0^y \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) dy \right| \ll \frac{\partial P}{\partial x} y, \tag{4.6}$$

or, after substituting the equilibrium distribution,

$$y\frac{d}{dx}\left(\log\frac{\partial P}{\partial x}\right) \ll \tfrac{3}{2}K_0^2. \tag{4.7}$$

For the power-law variation of free-stream velocity, $U_1 \propto x^a$, this is

$$\frac{y}{x} \ll \frac{3}{2}\frac{K_0^2}{1-2a}. \tag{4.8}$$

## 5. Initial stage of development

In the initial stage of development in a strong adverse pressure gradient, the outer layer behaves nearly as an inviscid fluid and the inner layer is an equilibrium layer whose motion is determined by the local pressure gradient and the wall stress. The mean velocity distribution at the beginning of the adverse pressure gradient may be written in the form

$$U = U_1(x_0)\,[1-\gamma f(\eta)], \tag{5.1}$$

where $\quad \gamma = \tau_0^{\frac{1}{2}}(x_0)/(KU_1(x_0)), \quad \eta = y/\delta, \quad \delta = \{\nu/KU_1(x_0)\}\exp(\gamma^{-1}-A).$

Within the equilibrium (constant-stress) layer,

$$f(\eta) = -\log\eta. \tag{5.2}$$

In the outer layer, total head remains constant along streamlines, and

$$P(x_0)+\tfrac{1}{2}U_0^2[1-\gamma f(\eta)]^2 = P(x)+\tfrac{1}{2}U^2(x,y) \tag{5.3}$$

if $(x_0, \eta\delta)$ and $(x, y)$ are on the same streamline. Within the inner layer,

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\left(\frac{\tau_0^{\frac{1}{2}}y}{\nu}\right)+A\right] \tag{5.4}$$

if $\tau_0 \gg (dP/dx)\,y$, and*

$$U = \frac{2}{K_0}\left(\frac{dP}{dx}y\right)^{\frac{1}{2}} \tag{5.5}$$

if $\tau_0 \ll (dP/dx)\,y$. Consider now the streamline which at $x$ divides the inner, equilibrium layer from the outer layer. Since the inner layer is an equilibrium layer, its distance from the wall must be small compared with the layer thickness at $x$ and therefore also at $x_0$. This dividing streamline lies in the constant-stress layer at $x_0$, and so we may use equation (5.2) for the velocity distribution function $f(\eta)$.

The condition that $(x_0, \eta_0\delta)$ and $(x, y_1)$ lie on the dividing streamline is

$$U_0\eta_0\delta(1-\gamma+\gamma\log\eta_0) = RU_0y_1\left(1-\gamma+\gamma\log R\frac{y_1}{\delta}\right) \tag{5.6}$$

or

$$= \frac{4}{3K_0}\left(\frac{dP}{dx}\right)^{\frac{1}{2}} y_1^{\frac{3}{2}}. \tag{5.7}$$

* The additive term $c(\nu\, dP/dx)^{\frac{1}{3}}$ in equation (4.5) is of order $U_1(U_1x/\nu)^{-\frac{1}{3}}$ and is negligible at Reynolds numbers over $10^6$.

The first joining condition is continuity of mean velocity, i.e. that the mean velocity given by (5.4) or by (5.5) satisfies equation (5.3):

$$P_0+\tfrac{1}{2}U_0^2(1+\gamma\log\eta_0)^2 = P+\tfrac{1}{2}R^2U_0^2\left[1+\gamma\log\left(R\frac{y_1}{\delta}\right)\right]^2 \tag{5.8}$$

or

$$= P+\frac{2}{K_0^2}\frac{dP}{dx}y_1. \tag{5.9}$$

The second joining condition is continuity of shear stress, and is, to a good approximation, that

$$K^2\gamma^2U_0^2 = R^2K^2\gamma^2U_1^2+\frac{dP}{dx}y_1. \tag{5.10}$$

The two sets of equations (5.6), (5.8), (5.10) and (5.7), (5.9), (5.10) describe respectively the initial development and the condition for attainment of zero wall stress. The second set may be solved explicitly, and this produces*

$$\exp\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}-1}{\gamma}\right]\times\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}}{\gamma}-1\right]x_0\frac{dc_p}{dx}$$
$$=\frac{8}{3}\frac{K^3}{K_0}\gamma^2\frac{x_0}{\delta} = \tfrac{8}{3}\beta K^3R_0\gamma^2\exp(A-\gamma^{-1}), \tag{5.11}$$

where $R_0 = (U_0x_0/\nu)$. If the origin of $x$ is so chosen that the boundary layer at $x_0$ has the same local friction coefficient as a constant-pressure layer that has developed for a distance $x_0$ (Townsend 1956*a*), we have

$$R_0 = [I_1-(2I_1+I_2)\gamma+2(I_1+I_2)\gamma^2]K^{-3}\gamma^{-2}\exp(\gamma^{-1}-A), \tag{5.12}$$

and then

$$\exp\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}-1}{\gamma}\right]\times\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}}{\gamma}-1\right]x_0\frac{dc_p}{dx}$$
$$= \tfrac{8}{3}\beta\gamma[I_1-(2I_1+I_2)\gamma+2(I_1+I_2)\gamma^2] = B. \tag{5.13}$$

This may be compared with Stratford's result, based on use of a power law for the initial velocity profile and use of continuity of velocity gradient as the second joining condition†

$$(2c_p)^{\frac{1}{4}(n-2)}\left(x\frac{dc_p}{dx}\right)^{\frac{1}{2}} = 1{\cdot}06\beta^{-1}(10^{-6}R_0)^{\frac{1}{10}}, \tag{5.14}$$

where $n = \log_{10}R_0$.

Stratford points out that a relation of this sort defines a pressure distribution in which a layer develops with continuously zero wall stress but without separa-

* Equations (5.2) and (5.10) are good approximations only if $\log\eta_0 < -3$, and the analysis is only valid if

$$c_p = (1+\gamma\log\eta_0)^2-2\beta^2\gamma^2 < (1-3\gamma)^2-2\beta^2\gamma^2.$$

For a Reynolds number of $10^6$, $\gamma \doteqdot 0{\cdot}1$ and the pressure recovery coefficient may not exceed 0·47.

† Stratford also makes an allowance for the effect of Reynolds stresses on the outer flow by assuming them to be unaffected by the pressure gradient. This refinement has been omitted in the derivation of (5.13).

tion, and he has succeeded in producing such a zero-stress layer by slight modification of the pressure distribution defined by equation (5.14) (Stratford 1959*b*). The theory of this paper gives the relation between pressure and distance as

$$\exp\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}-1}{\gamma}\right]\times\left\{\left[\frac{(c_p+2\beta^2\gamma^2)^{\frac{1}{2}}}{\gamma}-1\cdot5\right]^2+0\cdot75\right\}=\tfrac{1}{2}B\gamma^{-2}\frac{x-x_0}{x_0}, \quad (5.15)$$

and in figure 2 this pressure distribution is compared with Stratford's theoretical distribution which is an adequate representation of the experimental distribution over the expected range of validity of the theory. It is clear that both theories

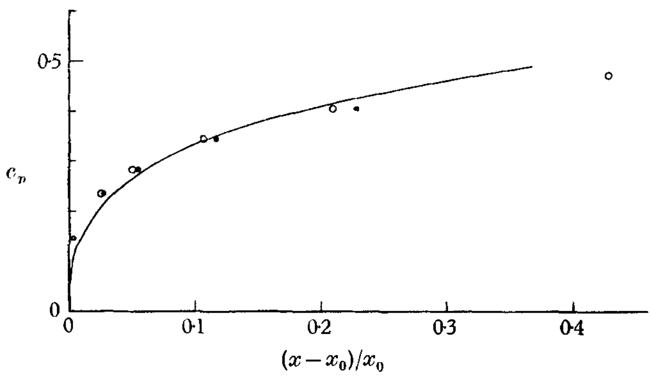


FIGURE 2. Comparison of predicted pressure distributions for development with continuously zero wall stress. Full line, Stratford ($K_0 = 0\cdot27$); ●, logarithmic theory ($K_0 = 0\cdot55$); ○, logarithmic theory ($K_0 = 0\cdot50$). $R_0 = 10^6, \gamma = 0\cdot10$.

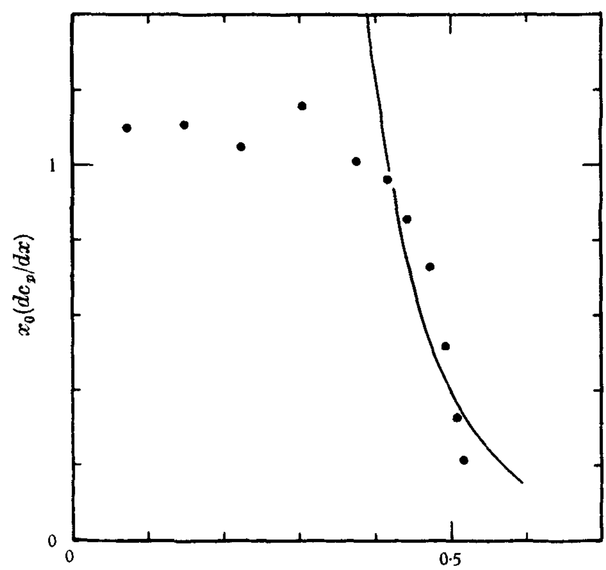


FIGURE 3. Comparison of the theoretical criterion for zero wall stress with measurements by Schubauer & Klebanoff (1951). (●, Experimental; full line, theory. $R_0 = 14\cdot3\times10^6$, $\gamma = 0\cdot075$, $K_0 = 0\cdot50$.)

are capable of describing the measurements for $c_p < 0{\cdot}40$ if a free choice of the value of the constant $K_0$ is allowed. Stratford used $K_0 = 0{\cdot}27$ compared with the values of 0·50 and 0·55 used in figure 2.

Equation (5.13) may be used to predict pressure-recovery coefficients at boundary-layer separation, i.e. zero wall stress, supposing the pressure distribution to be known and the adverse gradient to be so large that the approximations of the theory are valid. Of the available published material, only the layer studied by Schubauer & Klebanoff (1951) has an acceptably rapid rise of pressure, and this is compared with the rapid expansion theory in figure 3 by plotting $c_p$ against $x_0(dc_p/dx)$, as given by equation (5.13) and as observed. Zero stress is predicted at $c_p = 0{\cdot}42$, compared with observed separation at $c_p = 0{\cdot}50$, but the nature of the experimental pressure distribution is such that a comparatively small amount of interaction between the inner and outer layers would lead to a considerable delay in attaining zero wall stress. In its present form it is unlikely that equation (5.13) would provide accurate predictions of separation in practical problems, but the author believes that the theoretical basis is sound and that it should give accurate estimates of effects due to change of Reynolds number or initial boundary-layer characteristics.

## 6. The self-preserving flow with zero wall stress

It has been pointed out that, after a sufficient period of development, the structure of the whole layer will be modified by the pressure gradient, and then a simple description is possible only if the flow is self-preserving. Clauser (1956) has given details of self-preserving flows in adverse pressure gradients with finite wall stress, but Stratford has suggested that the flow with continuously zero wall stress, defined by equations (5.13) or (5.14), becomes self-preserving with a distribution of mean velocity

$$U = \frac{1}{2K_0}\left(\frac{dP}{dx}y\right)^{\frac{1}{2}} \quad \text{for} \quad y < \frac{4K_0^2 U_1^2}{dP/dx} \tag{6.1}$$

and a distribution of free-stream velocity

$$U_1 \propto (x - x_1)^{-0{\cdot}25}. \tag{6.2}$$

His experimental results suggest that this is a very rough description of the real behaviour, and a better one is easily obtained by using the theory of self-preserving boundary layers (Townsend 1956*b*). A turbulent flow is self-preserving if the distributions of mean velocity and of Reynolds stress may be expressed consistently in the forms

$$U = U_1 - u_0 f(y/l_0), \quad \overline{uv} = u_0^2 g(y/l_0), \tag{6.3}$$

where $f$ and $g$ are universal functions independent of $x$, and the scales $u_0$ and $l_0$ are functions only of $x$. The consistency may be tested by substitution in the averaged equations of motion, which leads to

$$-\frac{d(u_0 U_1)}{dx}f + \frac{u_0}{l_0}\frac{d(U_1 l_0)}{dx}\eta f' + u_0\frac{du_0}{dx}f^2 - \frac{u_0}{l_0}\frac{d(u_0 l_0)}{dx}f'\int_0^\eta f(s)\,ds + \frac{u_0^2}{l_0}g' = 0, \tag{6.4}$$

neglecting the viscous terms and terms involving normal Reynolds stresses. This is of self-preserving form if

$$u_0 \propto U_1, \quad \frac{dl_0}{dx} = \text{const.}, \quad \frac{l_0}{U_1}\frac{dU_1}{dx} = \text{const.},$$

and the scales may be defined as

$$u_0 = U_1, \quad l_0 = x \tag{6.5}$$

by choosing a suitable origin for $x$. It has been shown (Townsend 1956*a*) that exactly self-preserving flow with finite wall stress is possible only if $U_1 \propto (x_1 - x)^{-1}$, but another type of self-preserving flow is possible if the wall stress is zero. The reason for this is that the velocity distribution in the equilibrium layer may be written as

$$\frac{U}{U_1} = \frac{2}{K_0}\left(-\frac{x}{U_1}\frac{dU_1}{dx}\right)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{1}{2}} + C\left(-\frac{\nu}{U_1^2}\frac{dU_1}{dx}\right)^{\frac{1}{3}}, \tag{6.6}$$

which is a self-preserving form if $U_1 \propto (x - x_1)^a$ and if the additive term is negligibly small. The exponent $a$ is necessarily negative and depends on the shape of the velocity profile. A simple application of the equation for the momentum integral shows that

$$-a = \frac{I_a - I_b}{3I_a - 2I_b}, \tag{6.7}$$

where $$I_a = \int_0^\infty f(\eta)\,d\eta, \quad I_b = \int_0^\infty [f(\eta)]^2\,d\eta.$$

Experience with other self-preserving boundary layers has shown that an adequate approximation to the function $f(\eta)$ is obtained by assuming a sharp distinction between an inner equilibrium layer and an outer layer within which the effective eddy viscosity, $\nu_T = -\overline{uv}/(\partial U/\partial y)$, is independent of distance from the wall. Clauser (1956) has shown that the value of the eddy viscosity is such that

$$\frac{1}{\nu_T}\int_0^\infty (U_1 - U)\,dy = R_s, \tag{6.8}$$

where $R_s$ is nearly independent of pressure gradient (see also, Townsend 1956*b*). Assuming this, the mean velocity in the outer layer ($\eta > \eta_1$) is determined by the non-dimensional equation for $f(\eta)$

$$-2af + (a+1)\eta f' + af^2 - (a+1)f'\int_0^\eta f(s)\,ds + \frac{I_a}{R_s}f'' = 0, \tag{6.9}$$

and, in the inner layer, by $\quad f(\eta) = 1 - \dfrac{2}{K_0}(-a\eta)^{\frac{1}{2}}. \quad$ (6.10)

Equation (6.9) is a form of the Falkner–Skan equation, and solutions are required with the boundary conditions

$$f(0) = A, \quad f'(0) = 0, \quad f''(0) = \frac{R_s}{I_a}aA(2-A), \tag{6.11}$$

where $I_a$ and $a$ depend on the form of $f(\eta)$. An approximate solution of (6.9) is

$$f(\eta) = A\,e^{-\frac{1}{2}\alpha^2\eta^2}, \tag{6.12}$$

which satisfies the boundary conditions if

$$-\frac{\alpha^2 I_a}{aR_s} = 2 - A. \tag{6.13}$$

To make the problem determinate, the joining of the velocity distributions in the inner and outer layers must be specified by two independent conditions. Continuity of velocity requires that

$$A\, e^{-\frac{1}{2}\alpha^2\eta_1^2} = 1 - \frac{2}{K_0}(-a\eta_1)^{\frac{1}{2}}, \tag{6.14}$$

but, since the stress must satisfy equation (6.4), continuity of stress is assured for any value of $\eta_1$. In self-preserving layers developing in no pressure gradient, the junction is very nearly smooth, and so there is some justification in requiring continuity of velocity gradient, i.e. that

$$\alpha^2 A\eta_1 e^{-\frac{1}{2}\alpha^2\eta_1^2} = \frac{1}{K_0}\left(\frac{-a}{\eta_1}\right)^{\frac{1}{2}}. \tag{6.15}$$

If the values of $I_a$ and $I_b$ are approximated by assuming that the outer velocity distribution (6.12) extends all the way to $\eta = 0$, then

$$I_a = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{A}{\alpha}, \quad I_b = \frac{\pi^{\frac{1}{2}}}{2}\frac{A}{\alpha}, \tag{6.16}$$

and equations (6.7), (6.13), (6.14), (6.15) can be solved to give values of $A$, $a$, $\alpha\eta_1$, $\alpha K_0^2$ as functions of $K_0 R_s^{\frac{1}{2}}$. These quantities are tabulated in table 1.

| $K_0 R_s^{\frac{1}{2}}$ | $A$ | $-a$ | $\alpha\eta_1$ | $\alpha K_0^2$ |
|---|---|---|---|---|
| 3·02 | 0·696 | 0·252 | 0·6 | 3·46 |
| 3·70 | 0·756 | 0·241 | 0·5 | 4·32 |
| 4·19 | 0·787 | 0·235 | 0·45 | 5·08 |
| 4·32 | 0·795 | 0·234 | 0·44 | 5·26 |
| 4·87 | 0·821 | 0·228 | 0·4 | 6·19 |
| 7·16 | 0·886 | 0·214 | 0·3 | 11·02 |

TABLE 1

These predictions are compared with the experimental results of Stratford in figure 4, assuming a virtual origin for the self-preserving flow at $x/x_0 = 0{\cdot}94$. It is evident that the velocity distributions for $c_p = 0{\cdot}489$, $0{\cdot}624$, $0{\cdot}682$ are nearly of the same shape and that the layer thickness is increasing nearly linearly with distance from the effective origin. The actual velocity distributions are represented fairly well by the composite distribution for

$$A = 0{\cdot}795, \quad -a = 0{\cdot}234, \quad K_0 = 0{\cdot}50, \quad R_s = 74{\cdot}2,$$

except near the edges of the flow. The assumption of constant eddy viscosity always leads to the prediction of a velocity distribution that approaches the free-stream velocity less rapidly than the real one, and the discrepancy near the edge of the flow is not unexpected. The value of the exponent $a$ is less than the value suggested by Stratford ($-0{\cdot}25$), but it is in good agreement with the observed distribution of free-stream velocity (figure 5).

Two universal constants of uncertain magnitude appear in these results. The first is the flow constant $R_s$ which describes the moving equilibrium attained in the outer flow. The application of the theory to the experimental results gives $R_s = 74 \pm 4$, which is larger than the mean of the experimental values of Clauser.

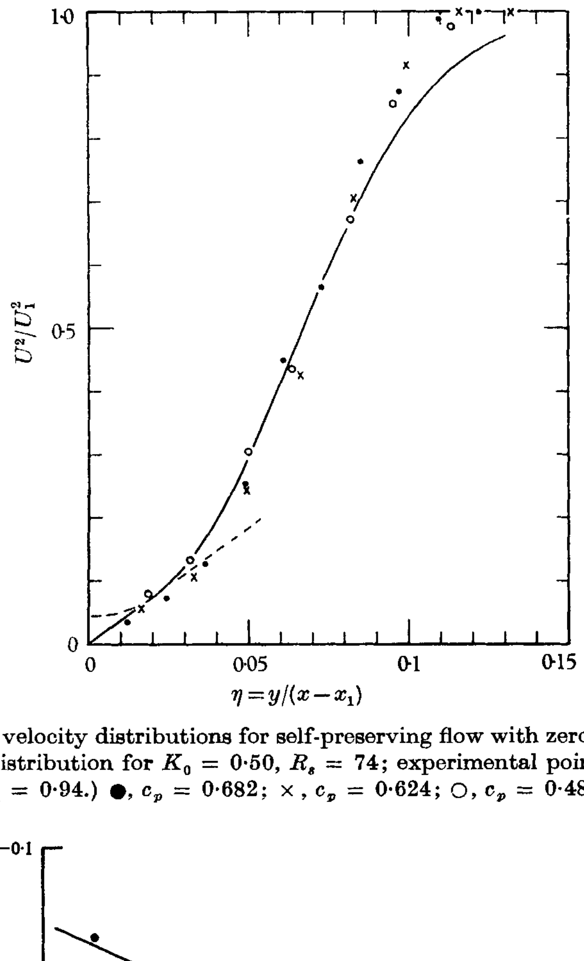


FIGURE 4. Mean velocity distributions for self-preserving flow with zero wall stress. (Full line, composite distribution for $K_0 = 0{\cdot}50$, $R_s = 74$; experimental points from Stratford (1959*b*) with $x/x_1 = 0{\cdot}94$.) ●, $c_p = 0{\cdot}682$; ×, $c_p = 0{\cdot}624$; ○, $c_p = 0{\cdot}489$.

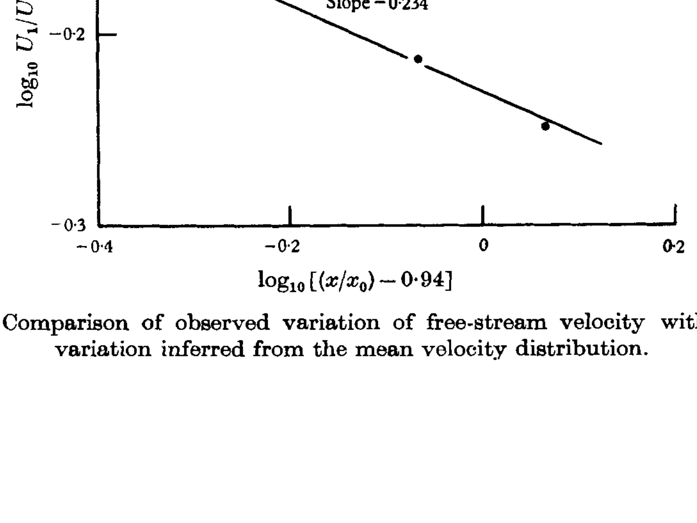


FIGURE 5. Comparison of observed variation of free-stream velocity with power-law variation inferred from the mean velocity distribution.

It is, however, close to the value deduced by a similar analysis of the constant-pressure boundary layer. The second constant is definitely rather larger than the Kármán constant, which is to be expected from a consideration of the effect of lateral transfer of turbulent energy on the level of turbulent intensity. The same value of $K_0$ will describe both the initial development and the self-preserving stage of the zero-stress layer. The conclusion is that $K_0 = 0{\cdot}50 \pm 0{\cdot}05$.

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